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STRESS ANALYSIS IN PLANE ORTHOTROPIC MATERIAL BY THE BOUNDARY ELEMENT METHOD

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Abstract—The boundary element method is a more suitable technique than the finite element method for problems having large stress gradients. One such problem is the stresses around mechanical fasteners. To solve a problem by the boundary element method requires a solution of an integral equation. The integrand of the integral equation is a product of a known Green's function and an unknown function. Unlike isotropic material, the plane orthotropic material can have three forms of Green's function depending upon the relationship of the four material constants. To solve the integral equations numerically, the unknown function is approximated by a piecewise continuous linear function. The boundary is approximated by a sum of straight line segments. The result of the two approximations is integrals over straight line segments, the integrand of which is a product of linear polynomials and one of the three Green's functions. These integrals are evaluated analytically. By exploiting the common features in the three forms of Green's function a very efficient algorithm can be designed. Numerical results are presented for a circular hole in an infinite medium and in a coupon. The results show good correlation with analytical results for all kinds of orthotropic materials.

1. INTRODUCTION

The boundary element method or BEM starts with the statement of the problem in terms of an integral equation. The integration in the integral equation is performed over the boundary. Thus, for numerical purposes when the discretization needs to be performed, it need be done only on the boundary. This procedure is in contrast to the finite element method in which discretization is done over the entire body. The net result of this difference is that the boundary element method generally requires less human and machine effort to solve a problem to the same level of accuracy.

The integrand in the integral equations is a product of a known Green's function and an unknown function. The Green's function by definition satisfies the differential equations exactly. Consequently, the solution of the stresses and the displacements satisfy the differential equations exactly. Once more this is in contrast with the finite element method where stresses are some weighted average over an element. Thus the resolution of high stress gradients like those near a pin hole in mechanical fastening problems, is much better by the boundary element method than by the finite element method.

The form of Green's function for isotropic materials is independent of the material properties. That is, the Green's function for isotropic materials is a linear combination of some singular functions. The material constants of isotropic material affect only the constants of the above mentioned linear combination. However, in orthotropic material, the nature and the form of the singular functions also change with the material constants. The dependence of the form of Green's function on the material constants has been known for a long time[1-5]. For plane, linear, orthotropic, elastic material there are three forms of Green's function depending upon the relationship of the four material constants. Two approaches have been used in the past to address the difficulty posed by the dependency of the form of Green's function on material properties. Reference [4] describes an algorithm using a single form of Green's function. This restricts the analysis to materials of a particular kind. The algorithms of Refs [2, 3, 5] are more general, but are based on complex variable representation of the Green's function. The advantage of complex variable representation is that a single expression describes all three forms of Green's function for a quantity. The disadvantage is that the algorithm uses a complex variable arithmatic, which usually requires more storage and computation time than a real variable arithmatic. In this work an alternative approach is taken. The Green's function was rederived using a Fourier transform[6]. Each form is expressed in terms of real variables. Care was taken in identifying the part of the Green's function the form of which would or would not change with the relationship of the material constants. By exploiting the common features of the three forms of Green's function an efficient algorithm has been designed.

The usual numerical approach in the evaluation of the integral is to approximate the unknown function in the integrand by a linear combination of known polynomials. This results in a linear expression in the unknown constants of the linear combination. The unknown constants are evaluated by satisfying the boundary condition at a finite number of points. The coefficients of the unknown constants are integrals containing the Green's function and the polynomials. These integrals are usually evaluated numerically. The advantage of numerical integration is that the order of the polynomial and the shape of the boundary can be of any complexity. However, in practice, only a cubic polynomial and a quadratic boundary shape have been used for an isotropic material. The disadvantage of numerical integration is that another source of error has been introduced into the process due to the approximation of the integrand for numerical integration. The singular nature of the Green's function further exacerbates this disadvantage, particularly when stresses need to be found near the boundary.

In this paper, the unknown function is assumed to vary piecewise linearly, while the boundary has been approximated by a sum of straight lines. The integrals are evaluated analytically for each form. In Ref. [5] the integrals were also evaluated analytically using a linear variation over straight line segments. The analytical expression, however, was described using complex variables. Usually the singularity contribution is written explicitly in the code and the remaining integral evaluated in the Cauchy sense. Thus the boundary element containing the singularity is usually treated differently from the rest of the boundary elements. In this work all boundary elements are treated in the same manner. This results in a simpler computer code. The correct value of the singularity contribution is obtained from the analytical expressions as described in Appendix B. The algorithm was tested on several geometries. Results for two cases are reported here. One corresponds to a circular hole in an infinite medium while the other problem corresponds to a coupon with a pin hole. Each geometry was considered for various relationships of material constants. An excellent correlation was found with analytical results (when available).

In Section 2, the formulation of the boundary value problem in terms of an integral equation is presented. Section 3 discusses the three forms of Green's function for plane orthotropic material. Problem discretization is presented in Section 4, and the numerical results in Section 5. In Appendix A, the mathematical expressions of the Green's function are reported. In Appendix B, the analytical expression for the integrals of the Green's function are given.

2. PROBLEM FORMULATION

Let σ_{ij} and e_{ij} represent the Cartesian stresses and strains at a point. It is assumed throughout this paper that the Cartesian axes and the material axes are parallel. The strains and stresses in an orthotropic material are related as

$$\begin{aligned} e_{xx} &= C_{11}\sigma_{xx} + C_{12}\sigma_{yy} \\ e_{yy} &= C_{12}\sigma_{xx} + C_{22}\sigma_{yy} \\ e_{xy} &= C_{33}\sigma_{xy} \end{aligned} \tag{1}$$

where the C's are material constants. Let $f_k(P)$ represent a point load in the direction k

applied at a point P. Let $H_{ijk}(Q, P)$ represent the Green's function relating the stresses at a point Q to a unit value of f_k at point P. By distributing the point load on the boundary B, we obtain an integral expression given by

$$\sigma_{ij}(Q) = \oint_B H_{ijk}(Q, P) f_k(P)\, \mathrm{d}s \quad i,j,k = x,y \tag{2}$$

sum over k

where s is the arc length measured from some arbitrary point to point P. To determine the unknown fictitious traction f_k, we need the boundary conditions. Only the traction boundary condition given below will be considered in this paper

$$\sigma_{ij}(Q) n_j(Q) = p_i(Q) \quad i,j = x,y \tag{3}$$

sum over j

where n_j are the direction cosines of the unit normal at point Q on the boundary and $p_i(Q)$ are the applied tractions.

Formally stated the solution procedure is as follows: substitute eqn (2) in eqn (3) and determine the function f_k. Once f_k is known, find stresses at any point Q using eqn (2).

3. THE GREEN'S FUNCTION

To find the Green's function, a two-dimensional infinite orthotropic plane is considered. The equilibrium equation, the compatibility equation, and the boundary condition at infinity are solved using the technique of Fourer transforms. The boundary condition at infinity is that the stresses and their first derivative go to zero.

Three forms of Green's function were found and are given in detail in Appendix A. These three forms of Green's function correspond to the nature of the roots of the following equation:

$$C_{11}\mu^4 + 2(C_{12}+C_{33})\mu^2 + C_{22} = 0. \tag{4}$$

The difference in eqn (4) from previous work[1–4] is due to the use of the tensor definition of strain in place of the engineering definition of strain.

The four roots of eqn (4) may be symbolically written as

$$\mu_1 = \bar{\imath}\lambda_1(\cos\delta_1 + \bar{\imath}\sin\delta_1) \tag{5a}$$

$$\mu_2 = -\bar{\imath}\lambda_1(\cos\delta_1 - \bar{\imath}\sin\delta_1) \tag{5b}$$

$$\mu_3 = \bar{\imath}\lambda_2(\cos\delta_2 + \bar{\imath}\sin\delta_2) \tag{5c}$$

$$\mu_4 = -\bar{\imath}\lambda_2(\cos\delta_2 - \bar{\imath}\sin\delta_2) \tag{5d}$$

where $\bar{\imath} = \sqrt{-1}$.

3.1. *Case* I: $((C_{12}+C_{33})/C_{11})^2 > (C_{22}/C_{11})$

For this case, $\delta_1 = \delta_2 = 0$, and

$$\lambda_{1:2} = \sqrt{\left(\frac{C_{12}+C_{33}}{C_{11}} \pm \sqrt{\left(\left(\frac{C_{12}+C_{33}}{C_{11}}\right)^2 - \frac{C_{22}}{C_{11}}\right)}\right)}. \tag{6}$$

The roots have no real part and are purely imaginary.

3.2. *Case* II: $((C_{12}+C_{33})/C_{11})^2 = (C_{22}/C_{11})$

For this case, $\delta_1 = \delta_2 = 0$, and

$$\lambda_1 = \lambda_2 = \sqrt{\left(\frac{C_{12}+C_{33}}{C_{11}}\right)}. \tag{7}$$

At first glance it would appear that this case is a degenerate case of Case I. This indeed is the case for terms T_1 and T_3 (eqns (A7a), (A7c) and (A9a), (A9c)) in the Green's function (A1). However, for the terms T_2 and T_4 (eqns (A7b), (A7d), and (A9b), (A9d)), there is a significant difference. Isotropic material belongs to this case and corresponds to $\lambda_1 = \lambda_2 = 1$.

Problems with material properties belonging to this case can also be solved by scaling the original geometry. For example, by scaling the y-coordinate by the factor given in eqn (7), the problem can be reduced to an isotropic case. A strategy used in Ref. [7] for solving an orthotropic plate problem by the boundary element method.

3.3. *Case* III: $((C_{12}+C_{33})/C_{11})^2 < (C_{22}/C_{11})$

For this case, $\delta_1 = \delta_2 = \delta$, and $\lambda_1 = \lambda_2 = \lambda$ where

$$\begin{aligned} \lambda \cos\delta &= \sqrt{\left(\frac{1}{2}\left(\sqrt{\left(\frac{C_{22}}{C_{11}}\right)}+\frac{C_{12}+C_{33}}{C_{11}}\right)\right)} \\ \lambda \sin\delta &= \sqrt{\left(\frac{1}{2}\left(\sqrt{\left(\frac{C_{22}}{C_{11}}\right)}-\frac{C_{12}+C_{33}}{C_{11}}\right)\right)}. \end{aligned} \tag{8}$$

Thus the roots in eqn (5) have a real and an imaginary part. From eqns (A7), (B12) and (A12), (B14), one sees that Cases I and III have very similar terms.

4. PROBLEM DISCRETIZATION

To solve the boundary valve problem given by eqns (2) and (3) numerically, we need to reduce the integral expression of eqn (2) to a linear algebraic expression. This is accomplished as follows.

4.1. *Assumption* 1

Assume that the fictitious traction f_k is linearly piecewise continuous over M segments of the boundary

$$f_k(P) = \frac{d_k^{(m+1)} - d_k^{(m)}}{S_{m+1} - S_m}(s - S_m) + d_k^{(m)} \tag{9}$$

$$S_m \leqslant s \leqslant S_1$$

$$m = 1, 2, \ldots, M$$

$$k = x, y$$

where S_m is the value of s at the mth node and $d_k^{(m)}$ are the unknown constants to be determined.

4.2. *Assumption* 2

Assume that the mth boundary segment can be represented by N_m straight line segments. This assumption does not introduce any more unknowns but permits a better approximation of the shape of the boundary. As shown in Ref. [8], the result is more accurate. Rewriting eqn (9) about the midpoint of each subdivision and substituting in eqn (2), we obtain

$$\sigma_{ij}(Q) = \sum_{m=1}^{M} \sum_{n=1}^{N_m} \left[\frac{d_k^{(m+1)} - d_k^{(m)}}{S_{m+1} - S_m} (\bar{s}_n - S_m) + d_k^{(m)} \right] M_{ijk}^{(0)}(Q, S_n) + \frac{d_k^{(m+1)} - d_k^{(m)}}{S_{m+1} - S_m} M_{ijk}^{(1)}(Q, S_n) \tag{10}$$

where $\bar{s}_n = (S_n + S_{n+1})/2$ and

$$M_{ijk}^{(q)}(Q, S_n) = \int_{S_n}^{S_{n+1}} (s - \bar{s}_n)^q H_{ijk}(Q, P)\, \mathrm{d}s. \tag{11}$$

Integrals $M_{ijk}^{(q)}$ can be found by substituting the appropriate formulas given in Appendixes A and B. For a closed boundary, continuity requires

$$d_k^{(M+1)} = d_k^{(1)}. \tag{12}$$

Thus there are M unknowns in each direction in eqn (10). To determine them we substitute eqn (10) in eqn (3), and satisfy the boundary condition at M points in a collocation sense. The result is a set of linear algebraic equations that can be solved for the unknown constants $d_k^{(m)}$. Once they are known, the stresses are evaluated from eqn (10).

The accuracy of the numerical solution to the algebraic equations is dependent upon the conditioning of the matrix. In Ref. [9], an algorithm is described which improves the conditioning of the matrix. It also makes the solution less sensitive to errors or changes in the input data. The algorithm of Ref. [9] is used in the present work.

5. RESULTS

The algorithm presented in this paper was tested on a number of geometries, loading and material properties. An excellent correlation with the analytical results was found in all cases. Results are presented for a geometry (Fig. 1) which can simulate two kinds of problems to be described later.

Each problem was solved for four kinds of material behavior which are shown in Table 1. All material constants were non-dimensionalized by the constant C_{11}. Case IIa represents the special orthotropic material requiring only three independent material constants. Case IIb represents the isotropic case.

5.1. *Problem 1: a circular hole in an infinite plane under uniaxial tension*

The geometry for this case was simulated by defining $D = 1$, $E = 50$, $H = 50$, $W = 100$ in Fig. 1. A uniform tension of $\sigma_{yy} = 1$ was applied in the y-direction. The analytical solution

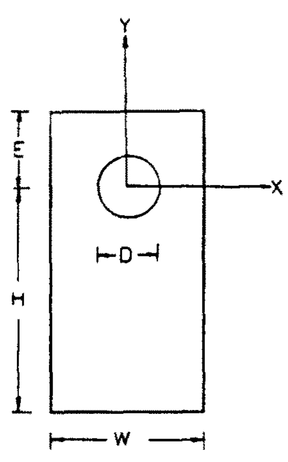

Fig. 1. Geometry of the test problem.

Table 1. Material properties

Material	C_{11}	C_{12}	C_{22}	C_{33}
Case I	1	−0.25	1	1.65
Case IIa	1	−0.25	1.96	1.65
Case IIb	1	−0.25	1	1.25
Case III	1	−0.25	1.96	1.25

to this problem is known[1] and is given below

$$K = \frac{\sigma_\theta}{\sigma_x} = \sqrt{\left(2\frac{C_{33}+C_{12}}{C_{22}} + 2\sqrt{\left(\frac{C_{11}}{C_{22}}\right)}\right)} + 1 \quad \text{at} \quad \theta = 0^\circ$$

where σ_θ is the tangential stress on the hole boundary and K is the stress concentration factor. The problem was solved with 168 unknowns ($M = 84$ in eqn (10)), and each problem took less than 6 min of CPU time on the IBM 4031 computer. Results for the stress concentration factor are shown in Table 2 for the various material constants. As can be seen there is very good agreement between the analytical and computed solution by the BEM.

5.2. *Problem* 2: *coupon under uniaxial tension*

For this problem the value of the variables shown in Fig. 1 were defined as $D = 1$, $E = 2.5$, $H = 7.5$, and $W = 5.0$. The problem was solved for the various material properties of Table 1 for two different sets of load cases.

5.2.1. *Problem* 2a: *traction free hole*. A uniform tension in the y-direction was applied at $y = E$ and $-H$. All other boundaries including the boundary of the hole are load free.

5.2.2. *Problem* 2b: *pin simulation*. A uniform tension was applied at $y = -H$. A normal traction varying as $\sin\theta$ was applied at the hole boundary for between 0 and 180°. The magnitude of the normal traction was chosen to produce static equilibrium. The traction on the hole is supposed to simulate a pin in mechanical fastening analysis[10].

The only solution that is known[4, 11] is for problem 2a for an isotropic material. Once more an excellent correlation is seen. Figures 2 and 3 show variation of normal stress σ_{yy} (or σ_θ) vs x at $\theta = 0$, for material Cases I and III, respectively. Results for material Cases IIa and IIb fall between the two curves shown and are not reported to avoid clutter in the diagram. By static equilibrium the area under these curves represent half the total force applied in the y-direction. This global equilibrium was used as an additional check on the solution. The computed and analytical (applied) value of the force in the y-direction is also reported in Table 2. Once more, good correlation is seen for these values.

6. CONCLUSIONS

This paper has demonstrated an efficient, accurate, and general algorithm for the solution of stresses in a two-dimensional orthotropic material having strong stress gradients. The paper underscores the importance of paying attention to the details of formulating the

Table 2. Numerical results

	Case I		Case IIa		Case IIb		Case III	
Problem	Analytical	Computed	Analytical	Computed	Analytical	Computed	Analytical	Computed
				Stress concentration factors				
1	3.191	3.053	2.690	2.604	3.000	2.894	2.565	2.604
2a	—	3.299	—	2.793	3.135	3.125	—	2.675
2b	—	6.454	—	5.073	—	6.005	—	4.734
				Force in the y-direction				
2a	5.000	4.993	5.000	5.009	5.000	5.004	5.000	5.009
2b	5.000	4.937	5.000	4.988	5.000	4.982	5.000	4.981

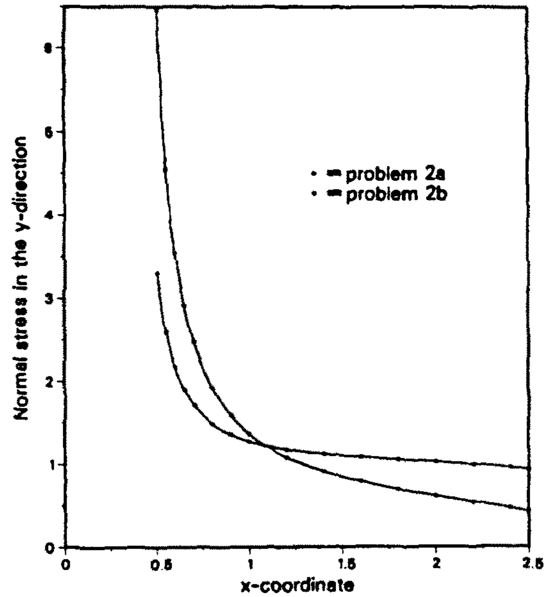

Fig. 2. Solution for Problem 2, Material Case I.

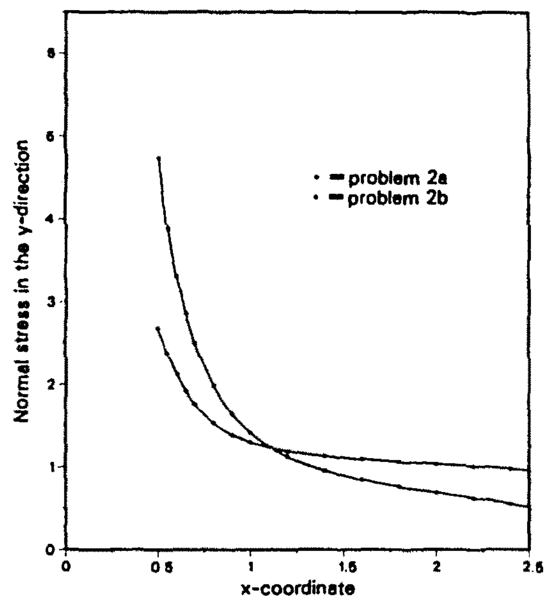

Fig. 3. Solution for Problem 2, Material Case III.

boundary value problem before embarking on the numerical aspects of the solution. Work on incorporating displacement and mixed boundary conditions is nearly complete and will be published in the near future.

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APPENDIX A: MATHEMATICAL EXPRESSIONS OF GREEN'S FUNCTION

The Green's function H_{ijk} has the following form for all material constant relations.

$$
\begin{aligned}
H_{xxx} &= E_{11}T_1 + E_{12}T_2 \\
H_{xxy} &= E_{21}T_3 + E_{22}T_4 \\
H_{yyx} &= E_{31}T_1 + E_{32}T_2 \\
H_{yyy} &= E_{41}T_3 + E_{42}T_4 \\
H_{xyx} &= E_{51}T_3 + E_{52}T_4 \\
H_{xyy} &= E_{61}T_1 + E_{62}T_2.
\end{aligned}
\tag{A1}
$$

The constant E_{ij} has the following form for all material constant relations:

$$
\begin{aligned}
E_{11} &= -(d_1 + d_3 - d_4)/(4\pi d_5) \\
E_{12} &= -(d_1 - d_3 - d_4)/(4\pi d_6) \\
E_{21} &= d_4/(4\pi) \\
E_{22} &= -(2d_2 - d_1 d_4)/(4\pi d_5 d_6) \\
E_{31} &= (1 + d_4/d_3)/(4\pi d_5) \\
E_{32} &= (1 - d_4/d_3)/(4\pi d_6) \\
E_{41} &= -1/4\pi \\
E_{42} &= (d_1 - 2d_4)/(4\pi d_5 d_6) \\
E_{51} &= E_{41} \\
E_{52} &= -E_{42} \\
E_{61} &= -d_3 E_{31} \\
E_{62} &= d_3 E_{32}
\end{aligned}
\tag{A2}
$$

where

$$
\begin{aligned}
d_1 &= 2(C_{12} + C_{33})/C_{11} \\
d_2 &= C_{22}/C_{11} \\
d_3 &= \sqrt{d_2} \\
d_4 &= C_{12}/C_{11}.
\end{aligned}
\tag{A3}
$$

Constants d_5 and d_6 as well as some of the T's in eqns (A1) change with the relationship between the material constants. Before defining them we define the following:

$$
\begin{aligned}
r_x &= x(Q) - x(P) \\
r_y &= y(Q) - y(P)
\end{aligned}
\tag{A4}
$$

where Q and P are field and source points as defined in eqn (2). Given below are the definitions of the remaining quantities for each kind of material.

Case 1: $(d_1/2)^2 > d_2$

$$d_5 = \lambda_1 + \lambda_2 \tag{A5a}$$

$$d_6 = \lambda_1 - \lambda_2 \tag{A5b}$$

where $\lambda_{1,2}$ are defined by eqn (6). Let

$$
\begin{aligned}
r_1^2 &= r_x^2 + \lambda_1^2 r_y^2 \\
r_2^2 &= r_x^2 + \lambda_2^2 r_y^2
\end{aligned}
\tag{A6}
$$

$$T_1 = \frac{r_x}{r_1^2} + \frac{r_x}{r_2^2} \qquad \text{(A7a)}$$

$$T_2 = \frac{r_x}{r_1^2} - \frac{r_x}{r_2^2} \qquad \text{(A7b)}$$

$$T_3 = \frac{\lambda_1 r_y}{r_1^2} + \frac{\lambda_2 r_y}{r_2^2} \qquad \text{(A7c)}$$

$$T_4 = \frac{\lambda_1 r_y}{r_1^2} - \frac{\lambda_2 r_y}{r_2^2}. \qquad \text{(A7d)}$$

Case II : $(d_1/2)^2 = d_2$. For this case $\lambda_1 = \lambda_2 = \lambda$ where λ is defined by eqn (7)

$$d_5 = \lambda_1 + \lambda_2 = 2\lambda \qquad \text{(A8a)}$$

$$d_6 = 1 \qquad \text{(A8b)}$$

$$T_1 = \frac{2r_x}{r_1^2} \qquad \text{(A9a)}$$

$$T_2 = -\frac{2\lambda_1 r_x r_y^2}{r_1^4} \qquad \text{(A9b)}$$

$$T_3 = \frac{2\lambda_1 r_y}{r_1^2} \qquad \text{(A9c)}$$

$$T_4 = \frac{r_x^2 r_y - \lambda_1^2 r_y^3}{r_1^4}. \qquad \text{(A9d)}$$

Note that the terms T_1 and T_3 in eqns (A9) would be obtained from eqns (A7) if we substitute $\lambda_1 = \lambda_2$. But the same is not true for the terms T_2 and T_4.

Case III : $(d_1/2)^2 < d_2$. For this case, λ_1 and λ_2 are complex quantities

$$\begin{aligned} d_5 &= 2\lambda \cos\delta \\ d_6 &= 2\lambda \sin\delta \end{aligned} \qquad \text{(A10)}$$

where λ and δ are defined in eqn (8). Define

$$\begin{aligned} \hat{r}_x(\delta) &= r_x - (\lambda \sin\delta) r_y \\ \hat{r}_y(\delta) &= \lambda \cos\delta\, r_y \\ \hat{r}^2(\delta) &= \hat{r}_x^2(\delta) + \hat{r}_y^2(\delta) \end{aligned} \qquad \text{(A11)}$$

$$\begin{aligned} T_1 &= \frac{\hat{r}_x(\delta)}{\hat{r}^2(\delta)} + \frac{\hat{r}_x(-\delta)}{\hat{r}^2(-\delta)} \\ T_2 &= -\frac{\hat{r}_y(\delta)}{\hat{r}^2(\delta)} + \frac{\hat{r}_x(-\delta)}{\hat{r}^2(-\delta)} \\ T_3 &= \frac{\hat{r}_y(\delta)}{\hat{r}^2(\delta)} + \frac{\hat{r}_y(-\delta)}{\hat{r}^2(-\delta)} \\ T_4 &= \frac{\hat{r}_x(\delta)}{\hat{r}^2(\delta)} - \frac{\hat{r}_x(-\delta)}{\hat{r}^2(-\delta)}. \end{aligned} \qquad \text{(A12)}$$

Note the similarity of form for the terms in eqns (A12) with those in eqns (A7).

APPENDIX B

B.1. *Analytical expressions of integrals of terms in the Green's function*

The functions $M_{ijk}^{(q)}$ in eqn (11) are integrals of the Green's function. Since the E_{ij} in the Green's function are constants (see eqns (A1)) we need to develop expressions for the integrals of T_1, T_2, T_3 and T_4 only.

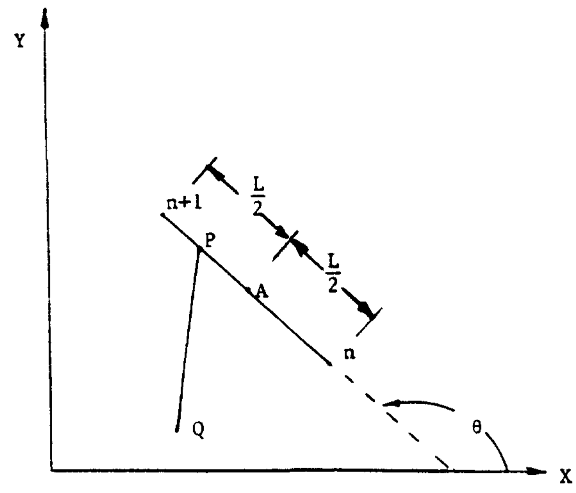

Fig. B1. Geometry of a line segment.

Let the length of the nth segment be L (see Fig. B1) and the angle which this segment makes with the x-axis be θ. Point A is the midpoint of the line segment with coordinates $(\bar{x}, \bar{y})$. For all values of λ_k and δ_k $(k = 1, 2)$ defined in eqns (6)–(8), we define the following quantities:

$$\begin{aligned} H_k^2 &= (\cos^2\theta + \lambda_k^2 \sin^2\theta + 2\lambda_k \sin\delta_k \sin\theta\cos\theta) \\ \cos\gamma_k &= (\cos\theta + \lambda_k \sin\delta_k \sin\theta)/H_k \\ \sin\gamma_k &= (\lambda_k \cos\delta_k \sin\theta)/H_k \end{aligned} \tag{B1}$$

$$\begin{aligned} C_k &= \{[(x-\bar{x}) + \lambda_k \sin\delta_k (y-\bar{y})]\cos\gamma_k + (y-\bar{y})\lambda_k\cos\delta_k\sin\gamma_k\}/H_k \\ D_k &= \{-[(x-\bar{x}) + \lambda_k\sin\delta_k(y-\bar{y})]\sin\gamma_k + (y-\bar{y})\lambda_k\cos\delta_k\cos\gamma_k\}/H_k \end{aligned} \tag{B2}$$

$$\begin{aligned} (R_k)^2_{n+1} &= \left(\frac{L}{2} - C_k\right)^2 + D_k^2 \\ (R_k)^2_n &= \left(\frac{L}{2} + C_k\right)^2 + D_k^2 \end{aligned} \tag{B3}$$

$$\begin{aligned} (\beta_k)_{n+1} &= \tan^{-1}\left(\frac{L/2 - C_k}{D_k}\right) \\ (\beta_k)_n &= \tan^{-1}\left(\frac{-(L/2 + C_k)}{D_k}\right) \end{aligned} \tag{B4}$$

$$\begin{aligned} \Delta L_k &= \log((R_k)_{n+1}/(R_k)_n) \\ \Delta\beta_k &= (\beta_k)_{n+1} - (\beta_k)_n \end{aligned} \tag{B5}$$

$$\begin{aligned} J_k^{(0)} &= -(\cos\gamma_k\,\Delta L_k + \sin\gamma_k\,\Delta\beta_k)/H_k \\ K_k^{(0)} &= -(\sin\gamma_k\,\Delta L_k - \cos\gamma_k\,\Delta\beta_k)/H_k \end{aligned} \tag{B6}$$

$$\begin{aligned} J_k^{(1)} &= -L\cos\gamma_k/H_k + C_kJ_k^{(0)} + D_kK_k^{(0)} \\ K_k^{(1)} &= -L\sin\gamma_k/H_k + C_kK_k^{(0)} - D_kJ_k^{(0)} \end{aligned} \tag{B7}$$

$$\begin{aligned} A_k &= D_k(L/2 - C_k)/(R_k)^2_{n+1} + D_k(L/2 + C_k)/(R_k)^2_n \\ B_k &= D_k^2/(R_k)^2_{n+1} - D_k^2/(R_k)^2_n \end{aligned} \tag{B8}$$

$$\begin{aligned} J_k^{(2)} &= (\cos 2\gamma_k\,J_k^{(0)} - \sin 2\gamma_k\,K_k^{(0)}) + 2(A_k\sin 3\gamma_k - B_k\cos 3\gamma_k) \\ K_k^{(2)} &= (\sin 2\gamma_k\,J_k^{(0)} + \cos 2\gamma_k\,K_k^{(0)}) - 2(A_k\cos 3\gamma_k + B_k\sin 3\gamma_k) \end{aligned} \tag{B9}$$

$$\begin{aligned} J_k^{(3)} &= -L\cos 3\gamma_k/H_k + 2D_k(\sin 2\gamma_k\,J_k^{(0)} + \cos 2\gamma_k\,K_k^{(0)}) + C_kJ_k^{(2)} + D_kK_k^{(2)} \\ K_k^{(3)} &= -L\sin 3\gamma_k/H_k - 2D_k(\cos 2\gamma_k\,J_k^{(0)} - \sin 2\gamma_k\,K_k^{(0)}) - D_kJ_k^{(2)} + C_kK_k^{(2)}. \end{aligned} \tag{B10}$$

For convenience of writing we define

$$I_j^{(q)} = \int_{S_n}^{S_{n+1}} (s-\bar{s}_n)^q T_j \quad q = 0, 1. \tag{B11}$$

The function $I_j^{(q)}$ is now defined for the various relations of material constants (see Sections 3.1–3.3).

Case I

$$\begin{aligned} I_1^{(q)} &= J_1^{(q)} + J_2^{(q)} \\ I_2^{(q)} &= J_1^{(q)} - J_2^{(q)} \\ I_3^{(q)} &= K_1^{(q)} + K_2^{(q)} \\ I_4^{(q)} &= K_1^{(q)} - K_2^{(q)}. \end{aligned} \tag{B12}$$

Case II

$$\begin{aligned} I_1^{(q)} &= 2J_1^{(q)} \\ I_2^{(q)} &= (J_3^{(q)} - J_1^{(q)})/2\lambda_1 \\ I_3^{(q)} &= 2K_1^{(q)} \\ I_4^{(q)} &= (K_3^{(q)} - K_1^{(q)})/2\lambda_1. \end{aligned} \tag{B13}$$

Case III

$$\begin{aligned} I_1^{(q)} &= J_1^{(q)} + J_2^{(q)} \\ I_2^{(q)} &= K_1^{(q)} - K_2^{(q)} \\ I_3^{(q)} &= K_1^{(q)} + K_2^{(q)} \\ I_4^{(q)} &= -J_1^{(q)} + J_2^{(q)}. \end{aligned} \tag{B14}$$

B.2. *Evaluation of singularity*

The Green's function $H_{ijk}(Q, P)$ is singular when Q and P coincide. This happens if stresses are to be evaluated on the boundary, for example when we try to satisfy the boundary conditions of eqn (3). The usual approach in BEM is to extract the singularity analytically and evaluate the integral in the Cauchy principle sense. This paper takes a different approach. Since we have analytical expressions, we can use them to give the correct value of the singularity. This is accomplished by choosing the point Q on the boundary as a midpoint of a small segment of length 2ε. That is in Fig. B1, Q and A are one and the same point. Thus $L/2 = \varepsilon$ and from eqns (B2)–(B4) we have $C_k = 0$, $D_k = 0$, $(R_k)_{n+1} = (R_k)_n = \varepsilon$, and $(\beta_k)_{n+1} = -(\beta_k)_n = \pi/2$. The computer code also calculates these values and when they are substituted in the formula, we obtain the correct value of the singularity contribution.